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SEMI-CLASSICAL LIMIT OF AN INFINITE DIMENSIONAL SYSTEM OF NONLINEAR SCHRÖDINGER EQUATIONS

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This contribution is devoted to Prof. Tai-Ping Liu on the occasion of his 70th birthday with all our thanks for his friendship and also for his contributions to our community with breakthroughs in Mathematical Science and unbreakable wisdom.

Abstract

We study the semi-classical limit of an infinite dimensional system of coupled nonlinear Schrödinger equations towards exact weak solutions of the Vlasov-Dirac-Benney equation, for initial data with analytical regularity in space. After specifying the right analytic extension of the problem and solutions, the proof relies on a suitable version of the Cauchy-Kowalewski Theorem and energy estimates in Hardy type spaces with convenient analytic norms. This contribution presents a detailed and probably optimal (with complete proofs) version of results announced in the more general setting in [1] and [2].

1. Introduction and Formal Derivations

The connection of the Schrödinger equation with a self-consistent potential to the Vlasov equation, via the Wigner or semi-classical limit, is at present very well documented. For instance for the Schrödinger−Poisson

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equation, i.e. with a self-consistent defocusing Coulomb type potential of the form,

$$V_\hbar(t,x)=\int_{\mathbb{R}^d}\frac{1}{|x-y|^{(d-2)}}|\psi_\hbar(t,y)|^2dy,$$

where the wave-function $\psi_\hbar$ is solution of the Schrödinger−Poisson equation, not only (with well adapted initial data) the problem is uniformly well posed but the convergence of the Wigner transform, on an arbitrary large time is also proven (cf. for instance [18] or [12]).

For the present contribution we consider a probability space $(\mathcal{M},d\sigma)$ and start with a family $\{\psi_\hbar(t,x,\sigma)\}_{\sigma\in\mathcal{M}}$, solution of the following cubic nonlinear Schrödinger equations

$$i\hbar\partial_t\psi_\hbar(t,x,\sigma)=-\frac{\hbar^2}{2}\Delta_x\psi_\hbar(t,x,\sigma)+\left(\int_{\mathcal{M}}|\psi_\hbar(t,x,\sigma)|^2d\sigma\right)\psi_\hbar(t,x,\sigma),\quad (1)$$

with $t\in\mathbb{R}^+$, $x\in\mathbb{R}^d$, $\sigma\in\mathcal{M}$ and some initial data $\{\psi_{\hbar 0}(x,\sigma)\}_{\sigma\in\mathcal{M}}$ which do not need to be specified right now. Given the potential

$$\mathcal{V}_\hbar(t,x)=\int_{\mathcal{M}}|\psi_\hbar(t,x,\sigma)|^2d\sigma,\quad (2)$$

the time-dependent equation

$$i\hbar\partial_t\theta_\hbar(t,x,\sigma)=-\frac{\hbar^2}{2}\Delta_x\theta_\hbar(t,x,\sigma)+\mathcal{V}_\hbar(t)\theta_\hbar(t,x,\sigma),$$

defines by the formula

$$\{\theta_\hbar(t,x,\sigma)\}_{\sigma\in\mathcal{M}}=\mathbb{U}_\hbar(t)\{\theta_0(x,\sigma)\}_{\sigma\in\mathcal{M}},$$

a family of unitary operators $\mathbb{U}_\hbar(t)$ acting in the space $L^\infty\big(\mathcal{M};L^2(0,T;L^2(\mathbb{R}^d_x))\big)$. Then we introduce the operator

$$K_\hbar(t,x,y)=\int_{\mathcal{M}}\psi_\hbar(t,x,\sigma)\otimes\overline{\psi_\hbar(t,y,\sigma)}d\sigma.$$

This operator is a solution of the Heisenberg-Von Neumann equation,

$$\frac{d}{dt}K_\hbar=-\frac{1}{i\hbar}[H_\hbar,K_\hbar]=-\frac{1}{i\hbar}\left[\frac{\delta\mathcal{E}_\hbar}{\delta K_\hbar},K_\hbar\right],\quad (3)$$

with the Hamiltonian

$$H_\hbar(t,x) = -\frac{\hbar^2}{2}\Delta_x + \mathcal{V}_\hbar(t,x),$$

and the total energy

$$\begin{aligned}\mathcal{E}_\hbar[K_\hbar](t) &= \mathrm{Trace}\left(-\frac{\hbar^2}{2}\Delta_x K_\hbar + \mathcal{V}_\hbar K_\hbar\right)\\ &= \int_{\mathbb{R}^d} dx \int_{\mathcal{M}} d\sigma \left(\frac{\hbar^2}{2}|\nabla_x \psi_\hbar(t,x,\sigma)|^2 + \mathcal{V}_\hbar(t,x)|\psi_\hbar(t,x,\sigma)|^2\right) < \infty,\end{aligned}$$

where the self-consistent potential $\mathcal{V}_\hbar$ is defined by equation (2) and $[\cdot,\cdot]$ denotes the commutator of operators. Formal solutions of the Schrödinger equations (1) and Heisenberg-Von Neumann equation (3) are respectively given by the implicit formulas (since unitary operators $\mathbb{U}_\hbar(t)$ depend on $\{\psi_\hbar(t)\}_{\sigma\in\mathcal{M}}$ through the potential $\mathcal{V}_\hbar(t)$)

$$\psi_\hbar(t,x,\sigma) = \mathbb{U}_\hbar(t)\psi_{\hbar 0}(x,\sigma) \quad \text{and} \quad K_\hbar(t,x,y) = \mathbb{U}_\hbar(t)K_{\hbar 0}(x,y)\mathbb{U}_\hbar(t)^*\,.$$

Eventually for the Wigner transform of the Heisenberg-Von Neumann equation, which involves the Weyl symbol $W_\hbar(t,x,v)$ with $x,\, v \in \mathbb{R}^d$, defined by the Wigner transform of $K_\hbar$,

$$W_\hbar(t,x,v) = \frac{1}{(2\pi)^d}\int_{\mathbb{R}^d} e^{-iy\cdot v} K_\hbar\Big(t, x+\frac{\hbar}{2}y, x-\frac{\hbar}{2}y\Big)dy,$$

one has from a formal viewpoint (i.e. assuming all sufficient conditions to pass to the limit) the following convergences as $\hbar \to 0$ (Wigner or semi-classical limit):

$$\begin{aligned}&W_\hbar(t,x,v) \longrightarrow W(t,x,v),\\ &\mathcal{E}_\hbar[K_\hbar] \longrightarrow \frac{1}{2}\int_{\mathbb{R}^d}\left(\int_{\mathbb{R}^d}|v|^2 W(t,x,v)dv + \left(\int_{\mathbb{R}^d} W(t,x,v)dv\right)^2\right)dx,\\ &\partial_t W + v\cdot\nabla_x W - \nabla_x\left(\int_{\mathbb{R}^d} W(t,x,w)dw\right)\cdot\nabla_v W = 0. \qquad (4)\end{aligned}$$

The last equation (4) dubbed as the "Vlasov−Dirac−Benney" equation (shortly V−D−B equation) is an avatar of the Vlasov−Poisson equation where the Coulomb potential is replaced by the Dirac mass. We refer the reader to [1, 2, 3, 5, 6, 16] for linear and nonlinear stability analysis, ill versus well posedness, and Hamiltonian structure of the V−D−B equation.

As observed by several authors [24, 13] it is always possible to write the solutions of the Vlasov equation - and especially for the V−D−B equation (4) - on the form

$$W(t,x,v)=\int_{\mathcal{M}}\rho(t,x,\sigma)\delta(v-u(t,x,\sigma))d\sigma. \tag{5}$$

These notations are consistent with the macroscopic definitions of density and momentum, according to the formulas,

$$\begin{aligned}\rho(t,x) &= \int_{\mathbb{R}^d} W(t,x,v)dv=\int_{\mathcal{M}}\rho(t,x,\sigma)d\sigma,\\ \rho(t,x)u(t,x) &= \int_{\mathbb{R}^d} vW(t,x,v)dv=\int_{\mathcal{M}}u(t,x,\sigma)\rho(t,x,\sigma)d\sigma.\end{aligned}$$

Such decomposition is not unique and depends in particular on the form of this decomposition at time $t=0$. Moreover a distribution function $W(t,x,v)$ given by (5) is a distributional solution of the V−D−B equation if and only if the functions $\rho(t,x,\sigma)$ and $u(t,x,\sigma)$ are solutions of the system

$$\begin{cases}\partial_t\rho(t,x,\sigma)+\nabla_x\cdot(\rho(t,x,\sigma)u(t,x,\sigma))=0,\\ \partial_t\left(\rho(t,x,\sigma)u(t,x,\sigma)\right)+\nabla_x\cdot(\rho(t,x,\sigma)u(t,x,\sigma)\otimes u(t,x,\sigma))\\ \;+\rho(t,x,\sigma)\nabla_x\displaystyle\int_{\mathcal{M}}\rho(t,x,\sigma)d\sigma=0.\end{cases} \tag{6}$$

In one space dimension with $(\mathcal{M},d\sigma)$ being respectively the interval $(0,1)$ and the Lebesgue measure, the system (6) turns out to be the Benney-Zakharov system

$$\begin{aligned}&\partial_t\rho(t,x,\sigma)+\partial_x(\rho(t,x,\sigma)u(t,x,\sigma))=0,\\ &\partial_t u(t,x,\sigma)+u(t,x,\sigma)\partial_x u(t,x,\sigma)+\partial_x\int_0^1\rho(t,x,\sigma)d\sigma=0,\end{aligned}$$

which has been derived by Zakharov from the original Benney equations [4] by using a Lagrangian parametrization (cf. [24]) as a model of water-waves for long waves. Hence the term “Benney” in the name of this Vlasov equation.

In order to consider mixed states as in [18], connect with the fluid representation formula (5) and in the mean time generalize to infinite dimensional system of coupled nonlinear Schrödinger equations the work of Grenier [14, 15] and Gerard [11] we assume that the functions $\psi_\hbar(t,x,\sigma)$, solutions to the Schrödinger equations (1), can be written as

$$\psi_\hbar(t,x,\sigma) = a_\hbar(t,x,\sigma)e^{i\frac{S_\hbar(t,x,\sigma)}{\hbar}}, \tag{7}$$

with $a_\hbar$ and $S_\hbar$ "uniformly regular" with respect to $\hbar$. Therefore for the Wigner transform one has

$$\begin{aligned}
&\lim_{\hbar\to 0} W_\hbar[K_\hbar](t,x,v)\\
&= \lim_{\hbar\to 0}\int_{\mathcal{M}} d\sigma \frac{1}{(2\pi)^d}\\
&\quad\times \int_{\mathbb{R}^d} e^{iv\cdot y} a_\hbar(t,x+\frac{\hbar}{2}y,\sigma)e^{i\frac{S_\hbar(t,x+\frac{\hbar}{2}y,\sigma)}{\hbar}}\overline{a_\hbar(t,x-\frac{\hbar}{2}y,\sigma)e^{-i\frac{S_\hbar(t,x-\frac{\hbar}{2}y,\sigma)}{\hbar}}}dy,\\
&= \int_{\mathcal{M}} |a(t,x,\sigma)|^2\delta(v-\nabla_x S(t,x,\sigma))d\sigma = \int_{\mathcal{M}}\rho(t,x,\sigma)\delta(v-u(t,x,\sigma))d\sigma.
\end{aligned}$$

where we have formally defined the limits $\rho = |a|^2 = \lim_{\hbar\to 0} a_\hbar\overline{a}_\hbar$ and $u = \nabla_x S = \lim_{\hbar\to 0}\nabla_x S_\hbar$. We observe that this formal limit satisfies the system

$$\begin{cases}
\partial_t\rho(t,x,\sigma) + \nabla_x\cdot(\rho(t,x,\sigma)u(t,x,\sigma)) = 0,\\
\partial_t u(t,x,\sigma) + u(t,x,\sigma)\cdot\nabla_x u(t,x,\sigma) + \nabla_x\int_{\mathcal{M}}\rho(t,x,\sigma)d\sigma = 0,
\end{cases} \tag{8}$$

which is the Benney system (6) or equivalently the V−D−B equation (4) with weak solutions of type (5). Moreover, on the other hand, with

$$\psi_\hbar(t,x,\sigma) = a_\hbar(t,x,\sigma)e^{i\frac{S_\hbar(t,x,\sigma)}{\hbar}}, \quad \text{and} \quad w_\hbar(t,x,\sigma) = \nabla_x S_\hbar(t,x,\sigma),$$

the Schrödinger system (1) is equivalent to the system

$$\begin{cases} \partial_t a_\hbar(t,x,\sigma) + w_\hbar(t,x,\sigma)\cdot\nabla_x a_\hbar(t,x,\sigma) + \dfrac{1}{2} a_\hbar(t,x,\sigma)\nabla_x\cdot w_\hbar(t,x,\sigma) \\ = \dfrac{i\hbar}{2}\Delta_x a_\hbar(t,x,\sigma), \\ \partial_t w_\hbar(t,x,\sigma) + w_\hbar(t,x,\sigma)\cdot\nabla_x w_\hbar(t,x,\sigma) + \nabla_x \displaystyle\int_{\mathcal{M}} a_\hbar(t,x,\sigma)\overline{a}_\hbar(t,x,\sigma)d\sigma \\ = 0. \end{cases} \tag{9}$$

Remark 1.1. The above representation is a variant both of the Madelung transform (where the amplitude $a_\hbar$ is taken real) and of the WKB method which is a Taylor expansion. As a consequence $a_\hbar(t,x,\sigma)$ does not remain real for $t\neq 0$, while $w_\hbar(t,x,\sigma)=\nabla_x S_\hbar(t,x,\sigma)$ remains.

To the best of our knowledge the above representation was first introduced by Chazarain [8, 9]. Later it was used by Grenier [14, 15] for the validation of the semi-classical limit for a genuine scalar nonlinear Schrödinger equation obtaining the following

Theorem 1.1 (Grenier [15]). *Let $s > d/2+2$, let $S^0(x)\in H^s(\mathbb{R}^d)$ and $a^0(x,\hbar)$ be a sequence of functions uniformly bounded in $H^s(\mathbb{R}^d)$. Then there exist $T>0$, and solutions*

$$\psi_\hbar(t,x) = a_\hbar(t,x)e^{i\frac{S_\hbar(t,x)}{\hbar}},$$

to the Cauchy problem

$$i\hbar\partial_t\psi_\hbar = -\frac{\hbar^2}{2}\Delta_x\psi_\hbar + |\psi_\hbar|^2\psi_\hbar, \qquad \psi_\hbar(0,x) = a^0(x,\hbar)e^{i\frac{S^0_\hbar(x)}{\hbar}}.$$

Moreover, $a_\hbar(t,x)$ and $S_\hbar(t,x)$ are bounded in $L^\infty\big(0,T;H^s(\mathbb{R}^d)\big)$ uniformly in $\hbar$.

From the Theorem 1.1 one deduces also the convergence (for $0<t<T$) of the quantities $a_\hbar\overline{a}_\hbar$ and $w_\hbar=\nabla_x S_\hbar$ respectively to $\rho(t,x)$ and $u(t,x)$ solutions of the system

$$\begin{cases} \partial_t\rho(t,x) + \nabla_x\cdot(\rho(t,x)u(t,x)) = 0, \\ \partial_t u(t,x) + u(t,x)\cdot\nabla_x u(t,x) + \nabla_x\rho(t,x) = 0. \end{cases} \tag{10}$$

To prove the Theorem 1.1, Grenier starts from the following system

$$\begin{cases} \partial_t w_\hbar(t,x) + w_\hbar(t,x)\nabla_x w_\hbar(t,x) + \nabla_x(\alpha_\hbar^2(t,x) + \beta_\hbar^2(t,x)) = 0, \\ \partial_t \alpha_\hbar(t,x) + w_\hbar(t,x)\cdot\nabla_x \alpha_\hbar(t,x) + \frac{1}{2}\alpha_\hbar(t,x)\nabla_x\cdot w_\hbar(t,x) = -\frac{\hbar}{2}\Delta_x\beta_\hbar(t,x), \\ \partial_t \beta_\hbar(t,x) + w_\hbar(t,x)\cdot\nabla_x \beta_\hbar(t,x) + \frac{1}{2}\beta_\hbar(t,x)\nabla_x\cdot w_\hbar(t,x) = \frac{\hbar}{2}\Delta_x\alpha_\hbar(t,x), \end{cases} \tag{11}$$

where $\alpha_\hbar(t,x)$ and $\beta_\hbar(t,x)$ denote respectively the real and imaginary part of $a_\hbar(t,x)$. He observes that this system can be symmetrized by a strictly positive matrix S and this will lead to the standard a priori estimates of hyperbolic systems of conservation laws [10, 19]. In fact the existence of such strictly positive symmetrizer is a consequence of the fact that the mass (i.e. with $\rho_\hbar(t,x) = \alpha_\hbar(t,x)^2 + \beta_\hbar(t,x)^2$) and the energy of the system,

$$\frac{1}{2}\int_{\mathbb{R}^d} \left(|w_\hbar(t,x)|^2 + \rho_\hbar(t,x)\right)\rho_\hbar(t,x)dx,$$

are a strictly convex invariants.

On the other hand, as observed in [1, 2, 3] for more general systems of the type (6), i.e. formal limit of mixed states solution of the the Heisenberg-Von Neumann equation, there may be no convex invariant. Therefore may exist (even in one space variable) initial data for which the Cauchy problem has no solution. This is due to the instantaneous appearance of exponential frequencies instabilities. Hence as this was done in the forerunner paper of [16] some type of control of analyticity is compulsory for a general theorem.

This is the object of the present contribution which mostly relies on two points. First the use of a refined version of the Cauchy-Kowalewski Theorem due to Safonov [22] and second the construction of well adapted spaces of analytical functions.

2. The Safonov Version of the Cauchy-Kowalewski Theorem

We recall that the Safonov Theorem concerns the equation

$$u(t) = \mathbb{T}u(t) = \int_0^t \mathscr{F}(\tau, u(\tau))d\tau, \tag{12}$$

where the mapping $\mathscr{F}$ is defined in a scale of Banach spaces $\mathbb{B}_\eta$ with

$$\text{for} \quad 0 < \eta' \leq \eta < \eta_0\,, \quad \mathbb{B}_\eta \subset \mathbb{B}_{\eta'}, \quad \|.\|_{\eta'} \leq \|.\|_\eta\,,$$

the following conditions, which in [22] are called "Assumptions 1.1."

(a) For some constants $\eta_0 > 0$, $r > 0$, $\lambda > 0$ and every pair of numbers η, η' such that $0 < \eta' < \eta < \eta_0$, $0 \leq t < \eta_0/\lambda$, the correspondence $(t, u) \mapsto \mathscr{F}(t, u)$ is a continuous mapping of

$$[0, \eta_0/\lambda) \times \{u \in \mathbb{B}_\eta : \|u\|_\eta < r\} \quad \text{into} \quad \mathbb{B}_{\eta'}\,.$$

(b) For any $0 < \eta' < \eta < \eta_0$, $0 \leq t < \eta_0/\lambda$, and for $u, v \in \mathbb{B}_\eta$ with $\|u\|_\eta < r\,, \|v\|_\eta < r$, we have

$$\|\mathscr{F}(t, u) - \mathscr{F}(t, v)\|_{\eta'} \leq \frac{C}{\eta - \eta'}\|u - v\|_\eta\,,$$

where C is a constant independent of η, η', t, u, v.

(c) $\mathscr{F}(t, 0)$ is a continuous function of $t \in [0, s_0/\lambda)$ with values in $\mathbb{B}_\eta$, $0 < \eta < \eta_0$, satisfying, with a fixed constant K

$$\|\mathscr{F}(t, 0)\|_\eta \leq \frac{K}{\eta_0 - \eta}, \quad 0 < \eta < \eta_0\,.$$

Then arises the

Theorem 2.1. *For any positive η_0, r, C and K, there is a positive constant λ_0 such that under the above assumptions with $\lambda > \lambda_0$, there exists a unique continuously differentiable function $u(t)$ with values in $\mathbb{B}_\eta$, $0 < \eta < \eta_0$, $\|u(t)\|_\eta < r$ which is defined for $0 \leq t < (\eta_0 - \eta)/\lambda$ and satisfies the problem (12).*

3. Notations and Definitions of Some Functional Spaces

In the d-dimensional complex euclidean space $\mathbb{C}^d$, we denote by $\mathbb{B}_{2d}(z, \delta)$ (resp. $\mathbb{S}_{2d-1}(z, \delta)$) the ball (resp. sphere) of $\mathbb{C}^d$ of center z and radius δ and by $\Pi_d(z, \delta)$ the cartesian product of circles $\mathbb{S}_1(z_i, \delta)$ for $i = 1, \ldots, d$. With η

being a given positive real number we define the tube T_η as the set of points of $\mathbb{C}^d$ such that

$$\mathrm{T}_\eta = \Big\{(x+iy) \in \mathbb{C}^d, \quad |y_j| \le \eta, \quad \text{for} \quad 1 \le j \le d\Big\}.$$

Then we denote by $\mathcal{A}(\mathrm{T}_\eta)$ the set of analytic functions on the tube T_η, and define the Hardy spaces $\mathrm{H}^p(\mathrm{T}_\eta)$, $1 \le p \le \infty$ by the formulas

$$\mathrm{H}^p(\mathrm{T}_\eta) = \Big\{ f \in \mathcal{A}(\mathrm{T}_\eta); \|f\|^p_{\mathrm{H}^p(\mathrm{T}_\eta)} = \sup_{\substack{|y_j| \le \eta \\ 1 \le j \le d}} \int_{\mathbb{R}^d} |f(x+iy)|^p dx < \infty \Big\}, 1 \le p < \infty,$$

$$\mathrm{H}^\infty(\mathrm{T}_\eta) = \Big\{ f \in \mathcal{A}(\mathrm{T}_\eta); \|f\|_{\mathrm{H}^\infty(\mathrm{T}_\eta)} = \sup_{z \in \mathrm{T}_\eta} |f(z)| < \infty \Big\}.$$

With $\Lambda_x = (1-\Delta_x)^{1/2}$ we define, for any non negative real number s and $p \in (1,\infty)$ the Hardy type spaces $\mathrm{H}^{s,p}_\eta$ by the formula

$$\begin{aligned} \mathrm{H}^{s,p}_\eta \;=\; \Big\{ & f(\cdot,\sigma) \in \mathrm{H}^p(\mathrm{T}_\eta) \text{ for a.e } \sigma \in \mathcal{M}; \\ & \|f\|_{\eta,s,p} = \|f\|_{\mathrm{H}^{s,p}_\eta} = \sup_{\sigma \in \mathcal{M}} \|\Lambda^s_x f(\cdot,\sigma)\|_{\mathrm{H}^p(\mathrm{T}_\eta)} < \infty \Big\}. \end{aligned}$$

A ball of radius r in $\mathrm{H}^{s,p}_\eta$ is denoted by

$$\mathcal{B}^{s,p}_\eta(r) = \Big\{ f \in \mathrm{H}^{s,p}_\eta ; \ \|f\|_{\eta,s,p} < r \Big\}.$$

Observe that, whenever s is an integer the expression

$$\|f\|^p_{\mathrm{H}^{s,p}_\eta} = \sup_{\sigma \in \mathcal{M}} \sum_{|\beta| \le s} \|\partial^\beta_x f\|^p_{\mathrm{H}^p(\mathrm{T}_\eta)},$$

provides an equivalent norm on $\mathrm{H}^{s,p}_\eta$, $p \in (1,\infty)$. Eventually for some fixed real numbers $\eta_0 > 0$, $\gamma \ge 0$ and $\lambda > 0$, we define the Banach spaces $\mathcal{H}^{s,p}_{\eta_0,\gamma,\lambda}$ by

$$\mathcal{H}^{s,p}_{\eta_0,\gamma,\lambda} = \Big\{ f(t) \in \mathrm{H}^{s,p}_\eta ; \ \|f\|^{(\gamma,\lambda)}_{\eta_0,s,p} = \sup_{0 < \eta + \lambda t < \eta_0} (\eta_0 - \eta - \lambda t)^\gamma \|f(t)\|_{\eta,s,p} < \infty \Big\},$$

with $\sup_{0<\eta+\lambda t<\eta_0}$ meaning $\sup_{0<\eta<\eta_0}\big(\sup_{0 \le t < (\eta_0-\eta)/\lambda}(\cdot)\big)$ or equivalently

$\sup_{0\leq t<\eta_0/\lambda}\left(\sup_{0<\eta<\eta_0-\lambda t}(\cdot)\right)$.

4. Main Theorem

With the above notations we state below the main contribution of this article as the following

Theorem 4.1. *Let $s>d/2$, $\eta_0>0$, $r>0$ and $\gamma\in[0,1)$ some positive real numbers independent of $\hbar$. Let us assume that initial data are such that*

$$\big(a_\hbar(0,x,\sigma),w_\hbar(0,x,\sigma)=\nabla_x S_\hbar(0,x,\sigma)\big)\in \mathrm{H}^{s,2}_{\eta_0}\cap\mathcal{B}^{s,2}_{\eta_0}(r). \tag{13}$$

Then there exists a positive real number $\lambda>0$ (depending on $s>d/2$, $\eta_0>0$, $r>0$ and $\gamma\in(0,1)$ but independent of $\hbar$) such that on the time interval $(0,\eta_0/\lambda)$ there exists a unique solution

$$\big(a_\hbar(t,x,\sigma),w_\hbar(t,x,\sigma)=\nabla_x S_\hbar(t,x,\sigma)\big)\in\mathcal{H}^{s,2}_{\eta_0,\gamma,\lambda}\cap\mathcal{B}^{s,2}_{\eta_0-\lambda t}(r),\quad \gamma\in[0,1),$$

of the problem

$$\begin{cases}\partial_t a_\hbar(t,x,\sigma)+w_\hbar(t,x,\sigma)\cdot\nabla_x a_\hbar(t,x,\sigma)+\dfrac{1}{2}a_\hbar(t,x,\sigma)\nabla_x\cdot w_\hbar(t,x,\sigma)\\ =\dfrac{i\hbar}{2}\Delta_x a_\hbar(t,x,\sigma),\\ \partial_t w_\hbar(t,x,\sigma)+w_\hbar(t,x,\sigma)\cdot\nabla_x w_\hbar(t,x,\sigma)+\nabla_x\displaystyle\int_{\mathcal{M}}a_\hbar(t,x,\sigma)\overline{a}_\hbar(t,x,\sigma)d\sigma=0,\end{cases}$$

with initial data (13). Moreover these solutions are uniformly bounded (with respect to $\hbar$) in $\mathcal{H}^{s,2}_{\eta_0,\gamma,\lambda}\cap\mathcal{B}^{s,2}_{\eta_0-\lambda t}(r)$.

Remark 4.1. The estimates being uniform with respect to $\hbar$ the Theorem 4.1 includes the case $\hbar=0$ and therefore it implies that the limit system (8) is well posed in $\mathcal{H}^{s,2}_{\eta_0,\gamma,\lambda}$ with solution

$$\mathcal{U}(t,x,\sigma)=\big(\rho(t,x,\sigma),u(t,x,\sigma)\big)\in\mathcal{B}^{s,2}_{\eta_0-\lambda t}(r),$$

for $t\in(0,\eta_0/\lambda)$ and initial data $\mathcal{U}(0,x,\sigma)=\big(\rho(0,x,\sigma),u(0,x,\sigma)\big)\in\mathcal{B}^{s,2}_{\eta_0}(r)$. A direct proof of this fact can be done. It follows the same line with minor simplifications.

On the other hand from the uniform estimates of the Theorem 4.1 one deduces that the solution

$$\mathcal{U}_\hbar(t,x,\sigma) = \big(|a_\hbar(t,x,\sigma)|^2, w_\hbar(t,x,\sigma) = \nabla_x S_\hbar(t,x,\sigma)\big),$$

arising from the system (9) converges as $\hbar \to 0$, to the solution $\mathcal{U}(t,x,\sigma) = \big(\rho(t,x,\sigma), u(t,x,\sigma)\big)$ of the limit system (8) in $\mathcal{H}^{s,2}_{\eta_0,\gamma,\lambda}$.

Hence the Theorem 4.1 gives at the same time the fact that the problem (8) is well posed in the class of well adapted Hardy spaces and that it is the semi-classical limit (in the same topology) of solutions of (9) or equivalently of (1).

Remark 4.2. In agreement with the representation formula (5), the Theorem 4.1 for the limit system (8) concerns (at variance with the Jabin-Nouri Theorem [16]) solutions which are analytic with respect to x and t but which can exhibit singularities in the v variable (sum of Dirac masses, etc ...).

5. Proof of the Theorem 4.1

We start with two following lemmas:

Lemma 5.1. *The operators* $\exp(\pm it\hbar\Delta_z) : \mathrm{H}^{s,2}_\eta \longrightarrow \mathrm{H}^{s,2}_\eta$ *are unitary in* $\mathrm{H}^{s,2}_\eta$ *for all* $\hbar \in \mathbb{R}$ *and* $t \in \mathbb{R}$, *and we get with* $s \geq 0$,

$$\| \exp(\pm it\hbar\Delta_z)\psi\|_{\eta,s,2} = \|\psi\|_{\eta,s,2}\,, \quad \hbar \in \mathbb{R},\ t \in \mathbb{R},\ \textit{and}\ \psi \in \mathrm{H}^{s,2}_\eta.$$

Proof. Using the holomorphic Fourier transform (cf. [21]) we observe that the Green function (i.e. the convolution kernel) associated to the operators $\exp(\pm it\hbar\Delta_z)$ is the same as for the real case in $\mathbb{R}^d$. Therefore the proof is the same than the real case in $\mathbb{R}^d$; for example we refer the reader to the Proposition 2.2.3 of Chapter 2 of [7]. □

Lemma 5.2. *Let* $f \in \mathrm{H}^p(\mathrm{T}_\eta)$ *with* $1 \leq p \leq \infty$, *then for all* $\eta' < \eta$ *we have,*

$$\|\partial_{z_j} f\|_{\mathrm{H}^p(\mathrm{T}_{\eta'})} \leq \frac{1}{|\eta-\eta'|}\|f\|_{\mathrm{H}^p(\mathrm{T}_\eta)}\,, \quad 1 \leq j \leq d.$$

Proof. Let T_η and $\mathrm{T}_{\eta'}$ two tubes of $\mathbb{C}^d$ such that $\eta' < \eta$. Let $z = x+iy \in \mathrm{T}_{\eta'}$ and $\mathbb{B}_{2d}(z,\epsilon)$ a ball of $\mathbb{C}^d$ such that $\epsilon \le |\eta-\eta'|$. Since $f \in \mathrm{H}^p(\mathrm{T}_\eta)$ with the Cauchy formula we have:

$$\partial_{z_j} f = \frac{1}{(2\pi i)^d}\int_{\Pi_d(z,\epsilon)} \frac{f(\xi)d\xi_1\cdots d\xi_d}{(\xi_1-z_1)\cdots(\xi_{j-1}-z_{j-1})(\xi_j-z_j)^2(\xi_{j+1}-z_{j+1})\cdots(\xi_d-z_d)},$$
$$= \frac{1}{(2\pi i)^d}\int_{\Pi_d(z,\epsilon)} \frac{f(\xi)d\xi}{(\xi_j-z_j)^2\prod_{l\neq j}(\xi_l-z_l)}.$$

For $1\le p<\infty$, using Hölder inequality with $1/p+1/q=1$ and the previous equality we obtain

$$\begin{aligned}\int_{\mathbb{R}^d}|\partial_{z_j}f|^p dx &\le \frac{1}{(2\pi)^{dp}}\int_{\mathbb{R}^d}dx\left(\int_{\Pi_d(z,\epsilon)}|f(\xi)|^p d\xi\right)\\ &\qquad\times\left(\int_{\Pi_d(z,\epsilon)}\frac{d\xi}{|\xi_j-z_j|^{2q}\prod_{l\neq j}|\xi_l-z_l|^q}\right)^{p/q},\\ &\le \frac{1}{(2\pi)^{dp}}\left(\frac{\epsilon^d}{\epsilon^{2q}\epsilon^{(d-1)q}}\right)^{p/q}\int_{\mathbb{R}^d}dx\int_{\Pi_d(z,\epsilon)}d\xi\,|f(\xi)|^p,\\ &\le \frac{1}{(2\pi)^{dp}}\frac{1}{\epsilon^p}\int_{\mathbb{R}^d}dx\frac{1}{\epsilon^d}\int_{\Pi_d(z,\epsilon)}d\xi\,|f(\xi)|^p.\end{aligned}$$

Using successively the mean value Theorem for analytic functions (cf. [21]) we get from the previous inequality the estimate

$$\begin{aligned}\int_{\mathbb{R}^d}|\partial_{z_j}f|^p dx &\le \frac{1}{(2\pi)^{dp}}\frac{1}{\epsilon^p}\int_{\mathbb{R}^d}|f(x+iy)|^p dx, \quad \forall\epsilon\le|\eta-\eta'|,\\ &\le \frac{1}{(2\pi)^{dp}}\frac{1}{|\eta-\eta'|^p}\int_{\mathbb{R}^d}|f(x+iy)|^p dx\,.\end{aligned}$$

This leads to the estimate of Lemma (5.2) for $1\le p<\infty$ after taking the supremun with respect to the variables y_j for $1\le j\le d$.

In the case where $p=\infty$ the Cauchy formula gives

$$\begin{aligned}|\partial_{z_j}f| &\le \frac{1}{(2\pi)^d}\|f(\cdot+iy)\|_{L^\infty(\mathbb{R}^d_x)}\int_{\Pi_d(z,\epsilon)}\frac{d\xi}{|\xi_j-z_j|^2\prod_{l\neq j}|\xi_l-z_l|},\\ &\le \frac{1}{(2\pi)^d}\frac{\epsilon^d}{\epsilon^2\epsilon^{d-1}}\|f(\cdot+iy)\|_{L^\infty(\mathbb{R}^d_x)}, \quad \forall\epsilon\le|\eta-\eta'|,\end{aligned}$$

$$\leq \frac{1}{(2\pi)^d}\frac{1}{|\eta-\eta'|}\|f(\cdot+iy)\|_{L^\infty(\mathbb{R}^d_x)}.$$

Taking the supremun with respect to the variables y_j for $1 \leq j \leq d$, this gives estimate of Lemma (5.2) for $p = \infty$. □

Now we observe that any function $x \mapsto f(x)$ defined for $x \in \mathbb{R}^d$, and which is the restriction of an analytic function $f(x+iy)$ defined on a tube T_η, can be represented (with the Paley-Wiener Theorem I of Chapter I in [20]) by the formula

$$f(x) = \int_{\mathbb{R}^d} e^{ix\cdot\xi}\hat{f}(\xi)d\xi,$$

with the Fourier transform $\hat{f}(\xi)$ decaying exponentially for $|\xi| \to \infty$. Hence the complex conjugate

$$\overline{f(x)} = \int_{\mathbb{R}^d} e^{-ix\cdot\xi}\overline{\hat{f}(\xi)}d\xi,$$

is also the Fourier transform of a function with the same exponential decay and therefore can be extended as analytic function in the complex domain according to the formula

$$f^*(x+iy) = \int_{\mathbb{R}^d} e^{-i(x+iy)\cdot\xi}\overline{\hat{f}(\xi)}d\xi.$$

Of course such extension does not coincide with the complex conjugate of $f(x+iy)$ for $y \neq 0$, but it belongs to the same class (in term of regularity) of analytical functions. With this remark in mind, one introduces the analytic extensions $a_\hbar(t,x+iy,\sigma)$, $a^*_\hbar(t,x+iy,\sigma)$, and $w_\hbar(t,x+iy,\sigma)$ of $a_\hbar(t,x,\sigma)$, $\overline{a}_\hbar(t,x,\sigma)$, and $w_\hbar(t,x,\sigma)$ respectively and write (with $z = x+iy \in \mathbb{C}^d$) the system (9) in the following equivalent form:

$$\begin{cases} \partial_t w_\hbar(t,z,\sigma)+w_\hbar(t,z,\sigma)\cdot\nabla_z w_\hbar(t,z,\sigma)+\nabla_z\displaystyle\int_{\mathcal{M}} a_\hbar(t,z,\sigma)a_\hbar^*(t,z,\sigma)d\sigma=0,\\ \partial_t a_\hbar(t,z,\sigma)+w_\hbar(t,z,\sigma)\cdot\nabla_z a_\hbar(t,z,\sigma)+\dfrac{1}{2}a_\hbar(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\\ \qquad\qquad =\dfrac{i\hbar}{2}\Delta_z a_\hbar(t,z,\sigma),\\ \partial_t a_\hbar^*(t,z,\sigma)+w_\hbar(t,z,\sigma)\cdot\nabla_z a_\hbar^*(t,z,\sigma)+\dfrac{1}{2}a_\hbar^*(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\\ \qquad\qquad =\dfrac{-i\hbar}{2}\Delta_z a_\hbar^*(t,z,\sigma). \end{cases} \tag{14}$$

With the notations

$$U_\hbar=\begin{pmatrix} w_\hbar(t,z,\sigma)\\ a_\hbar(t,z,\sigma)\\ a_\hbar^*(t,z,\sigma)\end{pmatrix},\qquad L_\hbar=\begin{pmatrix}0\\ \frac{i\hbar}{2}\Delta_z\\ -\frac{i\hbar}{2}\Delta_z\end{pmatrix},$$

and

$$F(U_\hbar)=-\begin{pmatrix} w_\hbar(t,z,\sigma)\cdot\nabla_z w_\hbar(t,z,\sigma)+\nabla_z\int_M a_\hbar(t,z,\sigma)a_\hbar^*(t,z,\sigma)d\sigma\\ w_\hbar(t,z,\sigma)\cdot\nabla_z a_\hbar(t,z,\sigma)+\frac{1}{2}a_\hbar(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\\ w_\hbar(t,z,\sigma)\cdot\nabla_z a_\hbar^*(t,z,\sigma)+\frac{1}{2}a_\hbar^*(t,z,\sigma)\nabla_z\cdot w_\hbar(t,z,\sigma)\end{pmatrix},$$

the system becomes

$$\partial_t U_\hbar=F(U_\hbar)+L_\hbar(U_\hbar),$$

which, using a Duhamel's formula, implies

$$U_\hbar(t)=e^{tL_\hbar}U_{\hbar 0}+\int_0^t e^{(t-\tau)L_\hbar}F(U_\hbar(\tau))d\tau. \tag{15}$$

Now using the following change of unknowns

$$\mathcal{u}_\hbar(t)=e^{-tL_\hbar}U_\hbar(t)-U_{\hbar 0}, \tag{16}$$

the Duhamel formula (15) becomes

$$\mathcal{u}_\hbar(t)=\mathbb{T}_\hbar\mathcal{u}_\hbar(t)=\int_0^t\mathscr{F}_\hbar\big(\tau,\mathcal{u}_\hbar(\tau)\big)d\tau=\int_0^t e^{-\tau L_\hbar}F\big(e^{\tau L_\hbar}(\mathcal{u}_\hbar(\tau)+U_{\hbar 0})\big)d\tau. \tag{17}$$

The formulation (17) of the problem (14) is now well suited to apply the Cauchy-Kowalewski Theorem 2.1 (cf. Section 2) whose proof consists in our functional framework to apply a Banach fixed-point Theorem to the nonlinear mapping $\mathbb{T}_\hbar$ in a certain subspace of $\mathcal{H}^{s,2}_{\eta_0,\gamma,\lambda}$ with $\gamma \in (0,1)$. Therefore it remains to verify assumptions (a) to (c) of the Section 2.

Let us start with assumption (a) of the Section 2. Let us fix $\eta_0 > 0$, $r > 0$ and the initial data $U_{\hbar 0}$ such that $\|U_{\hbar 0}\|_{\eta_0,s,2} < r$. Moreover let us assume $0 < \eta' < \eta < \eta_0$ and $\lambda > 0$. From the definition of the function $U \mapsto F(U)$, we observe that each component is a linear combination of two types of quadratic nonlinear terms, which are of the form $f \cdot \nabla_z g$ or $\int_{\mathcal{M}} d\sigma f \cdot \nabla_z g$. It is sufficient to study the first one since the second one deduces from the first one and the fact that we take the supremun norm in the variable σ over the set $\mathcal{M}$. Therefore we have to estimate terms of type $\partial_x^\beta(f \cdot \nabla_z g)$ with a spatial multi-index β such that $|\beta| \leq s$ where $s > d/2$ in the space $\mathrm{H}^{s,2}_{\eta'}$. Using Leibniz formula we get

$$\partial_x^\beta(f \cdot \nabla_z g) = \sum_{\alpha \leq \beta} \begin{pmatrix} \beta \\ \alpha \end{pmatrix} \partial_x^\alpha f \cdot \partial_x^{\beta-\alpha}(\nabla_z g).$$

Using the previous equality and the following classical bilinear estimate (e.g. cf. Proposition 3.6 of [23]): for $\mathfrak{f}$, $\mathfrak{g} \in H^s(\mathbb{R}^d_x) \cap L^\infty(\mathbb{R}^d_x)$, if $|\alpha| + |\beta| = s$ we have

$$\|(\partial_x^\alpha \mathfrak{f})(\partial_x^\beta \mathfrak{g})\|_{L^2(\mathbb{R}^d_x)} \leq C(s) \left(\|\mathfrak{f}\|_{L^\infty(\mathbb{R}^d_x)} \|\mathfrak{g}\|_{H^s(\mathbb{R}^d_x)} + \|\mathfrak{f}\|_{H^s(\mathbb{R}^d_x)} \|\mathfrak{g}\|_{L^\infty(\mathbb{R}^d_x)} \right),$$

we then obtain

$$\begin{aligned} &\|\partial_x^\beta(f \cdot \nabla_z g)\|_{L^2(\mathbb{R}^d_x)} \\ &\leq\ C(|\beta|) \left(\|f\|_{H^{|\beta|}(\mathbb{R}^d_x)} \|\nabla_z g\|_{L^\infty(\mathbb{R}^d_x)} + \|f\|_{L^\infty(\mathbb{R}^d_x)} \|\nabla_z g\|_{H^{|\beta|}(\mathbb{R}^d_x)} \right). \end{aligned}$$

Using the last estimate, the continuous Sobolev embedding $H^s(\mathbb{R}^d) \hookrightarrow L^\infty$ $(\mathbb{R}^d)$ if $s > d/2$ and the Lemma 5.2 with $p = 2, \infty$, we get

$$\begin{aligned} &\sup_{\sigma \in \mathcal{M}} \|\partial_x^\beta(f \cdot \nabla_z g)\|_{\mathrm{H}^2(\mathrm{T}_{\eta'})} \\ &\leq\ C(|\beta|) \left(\|f\|_{\eta',|\beta|,2} \sup_{\sigma \in \mathcal{M}} \|\nabla_z g\|_{\mathrm{H}^\infty(\mathrm{T}_{\eta'})}\ +\ \|f\|_{\eta',s,2} \|\nabla_z g\|_{\eta',|\beta|,2} \right), \\ &\leq\ \frac{C(|\beta|)}{|\eta - \eta'|} \left(\|f\|_{\eta,|\beta|,2} \|g\|_{\eta,s,2} + \|f\|_{\eta,s,2} \|g\|_{\eta,|\beta|,2} \right), \end{aligned}$$

$$\leq \frac{2C(|\beta|)}{|\eta-\eta'|}\|f\|_{\eta,s,2}\|g\|_{\eta,s,2},$$

which leads to $\|f\cdot\nabla_z g\|_{\eta',s,2} \leq C_\star(s)|\eta-\eta'|^{-1}\|f\|_{\eta,s,2}\|g\|_{\eta,s,2}$ and

$$\|F(U_\hbar)\|_{\eta',s,2} \leq \frac{C_\star(s)}{|\eta-\eta'|}\|U_\hbar\|^2_{\eta,s,2}, \quad s > d/2, \tag{18}$$

where $C_\star(s)$ denotes a set of constants which depend on s (but do not depend on $\hbar$) and are equivalent up to a multiplication by a pure numerical constant. Then the Lemma 5.1 implies that the operators $\exp(\pm tL_\hbar)$ are also unitary in $\mathrm{H}^{s,2}_\eta$. Therefore using estimate (18), the relation (16) and the isometry property of the operators $\exp(\pm tL_\hbar)$ we obtain

$$\begin{aligned}
&\sup_{\tau\in(0,\eta_0/\lambda)} \big\|\mathscr{F}_\hbar\big(\tau,u_\hbar\big)\big\|_{\eta',s,2}\\
&\leq \sup_{\tau\in(0,\eta_0/\lambda)} \sup_{u_\hbar\in\mathcal{B}^{s,2}_\eta(r)} \frac{2C_\star(s)}{|\eta-\eta'|}(\|u_\hbar\|^2_{\eta,s,2}+\|U_{\hbar0}\|^2_{\eta,s,2})\\
&\leq \frac{4r^2C_\star(s)}{|\eta-\eta'|} \leq \frac{C_\star(s,r)}{|\eta-\eta'|},
\end{aligned}$$

which means that $(t,u_\hbar)\mapsto\mathscr{F}_\hbar\big(\tau,u_\hbar\big)$ is a continuous mapping of $[0,\eta_0/\lambda)\times\mathcal{B}^{s,2}_\eta(r)$ into $\mathrm{H}^{s,2}_{\eta'}$ uniformly in $\hbar$, and concludes the point (a).

Let us check assumption (b) of the Section 2. Let us assume $0<\eta'<\eta<\eta_0$ and $0\leq t<\eta_0/\lambda$. We consider $u_\hbar,\ v_\hbar\in B^{s,2}_\eta(r)$. In the same way that we have obtained (18), since the mapping $U\mapsto F(U)$ is quadratic (and linear with respect to the first-order derivative) we obtain

$$\|F(U)-F(V)\|_{\eta',s,2} \leq \frac{C_\star(s)}{|\eta-\eta'|}\|U-V\|_{\eta,s,2}(\|U\|_{\eta,s,2}+\|V\|_{\eta,s,2}), \quad s>d/2. \tag{19}$$

Then using the isometry property of the operators $\exp(\pm tL_\hbar)$ and estimate (19) we have

$$\begin{aligned}
&\big\|\mathscr{F}_\hbar\big(t,u_\hbar\big)-\mathscr{F}_\hbar\big(t,v_\hbar\big)\big\|_{\eta',s,2}\\
&\leq \big\|F\big(e^{tL_\hbar}(u_\hbar(\tau)+U_{\hbar0})\big)-F\big(e^{tL_\hbar}(v_\hbar(\tau)+U_{\hbar0})\big)\big\|_{\eta',s,2},\\
&\leq \frac{C_\star(s)}{|\eta-\eta'|}\|u_\hbar-v_\hbar\|_{\eta,s,2}(2\|U_{\hbar0}\|_{\eta,s,2}+\|u_\hbar\|_{\eta,s,2}+\|v_\hbar\|_{\eta,s,2}),
\end{aligned}$$

$$\leq \frac{C_\star(s,r,\eta_0)}{|\eta-\eta'|}\,\|u_\hbar - \nu_\hbar\|_{\eta,s,2}\,,$$

which shows the uniform (both in time and $\hbar$ parameters) Lipschitz property of the application $u_\hbar \mapsto \mathscr{F}_\hbar\big(t, u_\hbar\big)$ and concludes the point (b).

Let us check assumption (c) of the Section 2. Let us assume $\eta_0 > 0$ and $\lambda > 0$. Since $[0, \eta_0/\lambda) \ni t \mapsto \exp(\pm t L_\hbar)$ is a strongly continuous group of unitary operators in $\mathrm{H}^{s,2}_\eta$ and $U \mapsto F(U)$ is a continuous mapping from $\mathrm{H}^{s,2}_{\eta_0}$ into $\mathrm{H}^{s,2}_\eta$ with $0 < \eta < \eta_0$, then $t \mapsto \mathscr{F}_\hbar(t, 0)$ is a continuous function of $t \in [0, \eta_0/\lambda)$ with value in $\mathrm{H}^{s,2}_\eta$ uniformly in $\hbar$. Moreover using the isometry property of the operators $\exp(\pm t L_\hbar)$ and estimate (18) we get

$$\|\mathscr{F}_\hbar(t,0)\|_{\eta,s,2} \leq \left\|F\big(e^{tL_\hbar}U_{\hbar 0}\big)\right\|_{\eta,s,2} \leq \frac{C_\star(s)}{|\eta_0-\eta|}\|U_{\hbar 0}\|^2_{\eta_0,s,2} \leq \frac{C_\star(s,r,\eta_0)}{|\eta_0-\eta|},$$

which concludes the point (c).

All assumptions of the Theorem 2.1 being satisfied it can be used to obtain the existence of the constant $\lambda > 0$ depending on $s > d/2$, $\eta_0 > 0$, $r > 0$ and $\gamma \in (0,1)$ but independent of $\hbar$ such that the conclusions of the Theorem 4.1 hold, which ends the proof.

Remark 5.1. The above proof is simpler than the forerunner result of Gérard [11], and it also provides an extension to mixed states as considered by Lions and Paul [18]. This is essentially due to the fact that the Theorem 2.1 (Safonov version of the Cauchy-Kowalewski theorem recalled above) is very well suited to the problem in the representation proposed by Chazarain [8, 9] and Grenier [14, 15].

6. Conclusion

As recalled above, in some situations (cf. Lions and Paul [18] for self-consistent Schrodinger equation with Coulomb Potential, Grenier [14, 15], or in the spirit of scattering theory by Jin, Levermore and McLaughlin [17] for the genuine nonlinear Schrodinger equation) the stability of the Wigner transform can be proven in Sobolev spaces of finite order. This is in full agreement with the fact that the limit Vlasov-type equation is a well-posed problem in the same type of spaces.

However, to justify the pertinence of the present contribution we observe that the V−D−B may lead to ill-posed problem in any spaces except in some analytical settings as in the Theorem 4.1. Similar pathology appears in the behavior of the Wigner (or semi classical) limit. Hence the introduction of theorem in the analytical setting is fully justified.

Moreover, the Theorem 4.1 can be applied to problems considered by Zakharov [24] in one space variable but involving also a (finite or infinite) system of coupled equations. As proven in [24] such system are integrable and this confers to the V−D−B equation a status of quasi-integrable equation with an infinity set of invariant quantities, limit of the corresponding invariants at the level of the Schrödinger equations. The above properties being in some sense algebraic, the proof of convergence with analyticity hypothesis seems perfectly well adapted to such considerations.

References

1. C. Bardos and N. Besse, The Cauchy problem for the Vlasov-Dirac-Benney equation and related issues in fluid mechanics and semi-classical limits, *Kin. Relat. Models*, **6** (2013), no.4, 893-917.

2. C. Bardos and N. Besse, Hamiltonian structure, fluid representation and stability for the Vlasov-Dirac-Benney equation, *Fields Inst. Commun.*, **75** (2015), 1-30.

3. C. Bardos and A. Nouri, A Vlasov equation with Dirac potential used in fusion plasmas, *J. Math. Phys.*, **53** (2012), no.11, 115621-115637.

4. D. J. Benney, Some properties of long nonlinear waves, *Stud. Appl. Math.*, **52** (1973), 45-50.

5. N. Besse, On the Waterbag Continuum, *Arch. Rat. Mech. Anal.*, **199** (2011), no.2, 453-491.

6. N. Besse, F. Berthelin, Y. Brenier and P. Bertrand, The multi-water-bag equations for collisionless kinetic modeling, *Kin. Relat. Models*, **2** (2009), no.1, 39-90.

7. T. Cazenave, "Semilinear Schrödinger equations", Lecture Notes **10**, Courant Institute of Mathematical Sciences, AMS, 2003.

8. J. Chazarain, Comportement semi classique d'un hamiltonien quantique, *Journées Equations aux dérivées partielles* **A17** (1979), 1-6.

9. J. Chazarain, Spectre d'un hamiltonien quantique et mécanique classique, *Comm. Partial Differential Equations*, **5** (1980), no.6, 595-644.

10. C. Dafermos, *Hyperbolic conservation Laws in continuum Physics*, Springer-Verlag, New York, 2000.

11. P. Gérard, Remarques sur l'analyse semi-classique de l'équation de Schrödinger non linéaire, Séminaire sur les Equations aux Dérivées Partielles, 1992-1993, Exp. no.XIII, 13 pp., Ecole Polytech., Palaiseau, 1993.

12. P. Gérard, P. Markowich, N. Mauser and F. Poupaud, Homogenization limits and Wigner transforms, *Comm. Pure Appl. Math.*, **50** (1997), no.4, 323-379.

13. E. Grenier, Oscillations in quasineutral plasmas, *Comm. Partial Differential Equations*, **21** (1996), no.3-4, 363-394.

14. E. Grenier, Limite semi-classique de l'équation de Schrödinger non linéaire en temps petit, *C. R. Acad. Sci. Paris Sér. I Math.*, **320** (1995), no.6, 691-694.

15. E. Grenier, Semiclassical limit of the nonlinear Schrödinger equation in small time, *Proc. Amer. Math. Soc.*, **126** (1998), no.2, 523-530.

16. P. E. Jabin, A. Nouri, Analytic solutions to a strongly nonlinear Vlasov, *C. R. Math. Acad. Sci. Paris*, **349** (2011), no.9-10, 541-546.

17. S. Jin, C. D., Levermore, D.W. McLaughlin, The semiclassical limit of the defocusing NLS hierarchy, *Comm. Pure Appl. Math.*, **52** (1999), no.5, 613-654.

18. P.-L. Lions, T. Paul, Sur les mesures de Wigner, *Rev. Mat. Iberoamericana*, **9** (1993), no.3, 553-618.

19. A. Majda, *Compressible Fluid Flow and Systems of Conservation Laws in Several Space Variables*, Applied Mathematical Sciences, 53. Springer-Verlag, New York, 1984.

20. R.C. Paley and N. Wiener, *Fourier Transforms in the Complex Plane*, AMS Vol 19, 1934.

21. W. Rudin, *Real and Complex Analysis*, Third edition, McGraw-Hill series in Higher Mathematics, 1987.

22. M. Safonov, The abstract Cauchy-Kovalevskaya theorem in a weighted Banach space, *Comm. Pure Appl. Math.*, **48** (1995), no.6, 629-637.

23. M. E. Taylor, *Partial Differential Equations III, Nonlinear Equations*, AMS Vol 117, Springer-Verlag, 1996.

24. V. E Zakharov, Benney equations and quasiclassical approximation in the inverse problem method, (Russian) *Funktsional. Anal. i Prilozhen.*, **14** (1980), no.2, 15-24.